Bessel's Inequality

Contents

Theorem

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space.

Let $\| \cdot \|$ be the inner product norm for $(V, \langle \cdot, \cdot \rangle)$.

Let $E = \{ e_n : n \in \mathbb{N} \}$ be a countably infinite orthonormal subset of V.

Then, for all $h \in V$:

$$\sum_{n = 1}^\infty |\langle h, e_n \rangle|^2 \le \|h\|^2$$

Corollary 1

Let E be a orthonormal subset of V.

Then, for each $h \in V$, the set:

$$\{ e \in E : \langle h, e \rangle \ne 0 \}$$

is countable.

Corollary 2

Let E be a orthonormal subset of V.

Then, for all $h \in V$:

$$\sum_{e \in E} |\langle h, e \rangle|^2 \le \|h\|^2$$

Proof

Note that for any natural number n we have, applying sesquilinearity of the inner product:

$$\begin{aligned} \left\| h - \sum_{k = 1}^n \langle h, e_k \rangle e_k \right\|^2 &= \left\langle h - \sum_{k = 1}^n \langle h, e_k \rangle e_k, h - \sum_{j = 1}^n \left\langle h, e_j \right\rangle e_j \right\rangle \\ &= \left\langle h, h - \sum_{j = 1}^n \left\langle h, e_j \right\rangle e_j \right\rangle - \left\langle \sum_{k = 1}^n \langle h, e_k \rangle e_k, h - \sum_{j = 1}^n \left\langle h, e_j \right\rangle e_j \right\rangle \\ &= \langle h, h \rangle - \left\langle h, \sum_{j = 1}^n \left\langle h, e_j \right\rangle e_j \right\rangle - \left\langle \sum_{k = 1}^n \langle h, e_k \rangle e_k, h \right\rangle + \left\langle \sum_{k = 1}^n \langle h, e_k \rangle e_k, \sum_{j = 1}^n \left\langle h, e_j \right\rangle \right. \end{aligned}$$

$$= \|h\|^2 - \left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle} + \left\| \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2$$

$$= \|h\|^2 - \left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle} + \sum_{k=1}^n \|\langle h, e_k \rangle e_k\|^2$$

$$= \|h\|^2 - \left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle} + \sum_{k=1}^n |\langle h, e_k \rangle|^2$$

We have:

$$\left\langle h, \sum_{j=1}^n \left\langle h, e_j \right\rangle e_j \right\rangle = \sum_{j=1}^n \left\langle h, \left\langle h, e_j \right\rangle e_j \right\rangle$$ sesquilinearity of inner product

$$= \sum_{j=1}^n \overline{\left\langle \left\langle h, e_j \right\rangle e_j, h \right\rangle}$$ conjugate symmetry of inner product

$$= \sum_{j=1}^n \overline{\left\langle e_j, h \right\rangle \left\langle h, e_j \right\rangle}$$

$$= \sum_{j=1}^n \left\langle h, e_j \right\rangle \overline{\left\langle h, e_j \right\rangle}$$ conjugate symmetry of inner product

$$= \sum_{j=1}^n \left| \left\langle h, e_j \right\rangle \right|^2$$ Product of Complex Number with Conjugate

Therefore:

$$\left\| h - \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 = \|h\|^2 - \sum_{j=1}^n \left| \left\langle h, e_j \right\rangle \right|^2 - \overline{\sum_{j=1}^n \left| \left\langle h, e_j \right\rangle \right|^2} + \sum_{k=1}^n |\langle h, e_k \rangle|^2$$

$$= \|h\|^2 - 2 \sum_{j=1}^n \left| \left\langle h, e_j \right\rangle \right|^2 + \sum_{k=1}^n |\langle h, e_k \rangle|^2$$ since $\left| \left\langle h, e_j \right\rangle \right|^2$ is real for each j, we have $\sum_{j=1}^n \left| \left\langle h, e_j \right\rangle \right|^2 \in \mathbb{R}$

$$= \|h\|^2 - \sum_{k=1}^n |\langle h, e_k \rangle|^2$$

Since:

$$\left\| h - \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 \geq 0$$

we have:

$$\sum_{k=1}^n |\langle h, e_k \rangle|^2 \leq \|h\|^2$$

Since:

$|\langle h, e_k \rangle|^2 \geq 0$ for each k

we have that:

the sequence $\left\langle \sum_{k=1}^n |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}}$ is increasing.

So:

$$\text{the sequence } \left\langle \sum_{k=1}^{n} |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}} \text{ is bounded and increasing.}$$

So from Monotone Convergence Theorem (Real Analysis): Increasing Sequence, we have that:

$$\text{the sequence } \left\langle \sum_{k=1}^{n} |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}} \text{ converges.}$$

Since:

$$\sum_{k=1}^{n} |\langle h, e_k \rangle|^2 \le \|h\|^2 \text{ for each } n$$

we then have from Limits Preserve Inequalities:

$$\|h\|^2 \ge \lim_{n \to \infty} \sum_{k=1}^{n} |\langle h, e_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle h, e_k \rangle|^2$$

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Source of Name

This entry was named for Friedrich Wilhelm Bessel.

Sources

- 1990: John B. Conway: *A Course in Functional Analysis* (2nd ed.) ... (previous) ... (next): I Hilbert Spaces: $\S 4$. Orthonormal Sets of Vectors and Bases: 4.8 Bessel's Inequality