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Euler's Rediscovery of e With Instructor Notes

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1 Introduction

The famous constant e is used in countless applications across many fields of mathematics, and resurfaces periodically in the evolution of mathematics. In 1683, Jacob Bernoulli (1655–1705) essentially found e while studying compound interest and evaluating the sequence $(1 + 1/j)^j$ as $j \to \infty$. By 1697, his brother Johann Bernoulli (1667–1748) was working with the calculus of exponentials [Bernoulli, 1697]. However, a full understanding was missing.

The connection between logarithms and exponential functions was still not well understood, and mathematicians couldn't agree on how to define logarithms of negative numbers. Leonhard Euler (1707–1783) would later clear up the controversy surrounding logarithms of negative numbers, and clarify the idea of a logarithmic *function*, in a paper written for the Berlin Academy of Sciences [Euler, 1749].

In 1748, Euler published one of his most influential works, *Introductio in Analysin Infinitorum* (*Introduction to Analysis of the Infinite*), [Euler, 1748]. Prior to the early nineteenth century, the word "analysis" that appears in the title generally referred to what we would today call "calculus." In the *Introductio*, Euler introduced the general function concept as the foundation of calculus.[1] While it was explicitly written as a "precalculus textbook" to prepare his readers for the study of calculus, Euler nevertheless emphasized the use of infinite series throughout the *Introductio*.[2] In Chapter VI, he discussed logarithms for various bases and their properties. Logarithms were well known in Euler's day, and tables of logarithms for base 10 and other bases had been compiled, as no scientific calculators were available at that time.[3] Euler also examined exponential and logarithmic functions in Chapter VII, especially as infinite series. In this project, we are particularly interested in how e appeared naturally in Euler's development of these functions.

2 Euler's Definition of e

Part of Euler's challenge in working with logarithmic functions was to find a logarithmic base a for which infinite series expansions are convenient. It is here that Euler derived e, both as the

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[1]Prior to Euler's work, calculus was viewed as the study of *curves*, rather than the study of *functions*.

[2]Infinite series were considered a prerequisite for the study of calculus even prior to Euler's work, and were used extensively by Newton, Leibniz and other seventeenth century mathematicians.

[3]As recently as the 1970s, most students used tables rather than calculators to find logarithms.

limiting value of $(1+1/j)^j$ and as the infinite series $1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots$. As was common in his day, Euler worked with *infinitely small and large numbers,* a practice that has largely been abandoned with the modern definition of limit. Nevertheless, Euler used his infinitely small and large numbers with great skill, as we shall see.[4]

Section 114. Since $a^0 = 1$, and as the exponent of a increases, the value of the power increases at the same time, provided a is a number greater than unity, it follows that if the exponent exceeds zero by an infinitely small amount, then the power itself will also exceed 1 by an infinitely small amount. Letting ω be an infinitely small number ..., then $a^\omega = 1+\psi$, where ψ is also an infinitely small number .... Letting $\psi = k\omega$, so that $a^\omega = 1+k\omega$; and taking a as the base of the logarithm, we will have $\omega = \log(1+k\omega)$.

EXAMPLE

That it may be more clearly seen how the number k depends on the base a, let us suppose that $a = 10$; and let us seek from the vulgar tables[5] the logarithm of a number which exceeds 1 by a small amount, say $1+\frac{1}{1000000}$, so that $k\omega = \frac{1}{1000000}$; this will be $\log\left(1+\frac{1}{1000000}\right) = \log\left(\frac{1000001}{1000000}\right) = 0.00000043429 = \omega$. Hence, $k\omega = 0.0000010000$ and $k = \frac{100000}{43429} = 2.30258$; from this it is clear that k is a finite number depending on the value of the base a. For if another number is set as the base a, then the logarithm [base a] of the same number $1+k\omega$ will have a given ratio with the former,[6] so that at the same time a different value of the variable k will appear.

Task 1 To gain some visual insight into what Euler was doing, plot $y = a^\omega$ and $y = 1+k\omega$ for $a = 10$ and $k = \frac{100000}{43429} \approx 2.30258$. Euler claimed these quantities a^ω and $1+k\omega$ should be identical for "infinitely small" ω. Would changing the k value to something else, say -3, change anything about your plot and this claim?

Task 2 Use Euler's ideas and a scientific calculator to estimate k for $a = 2$. Get a visual check by plotting $y = 2^\omega$ and $y = 1+k\omega$ together.

Euler was interested in finding an a value for which exponential and logarithmic expansions are nice and easy to work with. He derived a series expansion in his Section 115.

[4] All translations of Euler excerpts in this project were prepared by Janet Heine Barnett, 2022.

[5] These are tables of values for the common, or base 10, logarithmic function. Such tables would have an entry that exceeds 1 by "a small amount" for the specified base.

[6] *Translator's Note*: By "the former" Euler meant $\log_{10}(1+k\omega)$. In essence, he was referencing here the fact that, for any given base value a, the ratio $\frac{\log_a(x)}{\log_{10}(x)}$ is constant for all x.

Section 115. Since $a^\omega = 1 + k\omega$, we will have $a^{j\omega} = (1 + k\omega)^j$, for whatever number is substituted for j.[7] It will therefore follow that

$$a^{j\omega} = 1 + \frac{j}{1}k\omega + \frac{j\,(j-1)}{1\cdot 2}k^2\omega^2 + \frac{j\,(j-1)\,(j-2)}{1\cdot 2\cdot 3}k^3\omega^3 + \ldots \tag{1}$$

Therefore, if we set $j = \frac{z}{\omega}$, and z denotes an arbitrary finite number, then because ω is an infinitely small number, j will be an infinitely large number; hence $\omega = \frac{z}{j}$, so that ω is a fraction having an infinite denominator, and therefore infinitely small, as is assumed. Therefore substituting $\frac{z}{j}$ in place of ω, we will have

$$\begin{aligned} a^z = (1 + kz/j)^j &= 1 + \frac{1}{1}kz + \frac{1\,(j-1)}{1\cdot 2\cdot j}k^2z^2 \\ &\quad + \frac{1\,(j-1)\,(j-2)}{1\cdot 2j\cdot 3j}k^3z^3 + \frac{1\,(j-1)\,(j-2)\,(j-3)}{1\cdot 2j\cdot 3j\cdot 4j}k^4z^4 + \ldots, \end{aligned} \tag{2}$$

where this equation will be true when an infinitely large number is substituted for j. But then k is a number defined by a, as we have just seen.

We would like to capture the spirit of Euler's ideas but put his work on modern foundations by avoiding infinitely small and large numbers.

Task 3 Assume $a > 1$ and ω is a small, positive finite number defined by $a^\omega = 1 + \psi$ and $\psi = k\omega$.

(a) What theorem was Euler using to obtain (1)? For what ψ values is this series known to converge?

(b) Verify the algebraic details needed to obtain (1) from this theorem.

Task 4 Assume $a > 1$ and ω is a small, positive finite number defined by $a^\omega = 1 + \psi$ and $\psi = k\omega$, and $j = z/\omega$.

(a) What is the general n^{th} term in the series (2)?

(b) Verify the algebraic details needed to obtain (2) from $j = z/\omega$ and (1).

Euler next used his infinitely large numbers to produce an infinite series expression for his ideal logarithm base a. At this point in his book, Euler set $z = 1$ to find his special value for a.

Section 116. Now since j is an infinitely large number, $\dfrac{j-1}{j} = 1$; for it is clear that the greater the number substituted for j, the more the actual value of the fraction $\dfrac{j-1}{j}$

[7] *Translator's Note:* In this project, we use the letter j where Euler instead used the letter i, since the latter is open to misinterpretation as $\sqrt{-1}$ by today's readers. Although Euler was responsible for introducing the notational convention $i = \sqrt{-1}$, he did not start using it himself until 1777.

will approach unity, hence if j is a number greater than any assignable quantity, the fraction $\frac{j-1}{j}$ itself will become equal to 1. For a similar reason we will have $\frac{j-2}{j} = 1$; $\frac{j-3}{j} = 1$; and so on; hence it follows that $\frac{j-1}{2j} = \frac{1}{2}$; $\frac{j-2}{3j} = \frac{1}{3}$; $\frac{j-3}{4j} = \frac{1}{4}$; and so on. Therefore, with these values substituted [into equation (2)], we will have $a^z = 1 + \frac{kz}{1} + \frac{k^2z^2}{1 \cdot 2} + \frac{k^3z^3}{1 \cdot 2 \cdot 3} + \frac{k^4z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \ldots$ to infinity. And this equation at the same time illustrates the relationship between the numbers a and k, for if $z = 1$, we will have

$$a = 1 + \frac{k}{1} + \frac{k^2}{1 \cdot 2} + \frac{k^3}{1 \cdot 2 \cdot 3} + \cdots, \tag{3}$$

. . . .

Section 122. Since in order to establish a system of logarithms it is possible to take the base a at will, it may be assumed that [a is chosen] so that $k = 1$. Let us therefore suppose that $k = 1$, and by the series found above (Section 116),

$$a = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots, \tag{4}$$

whose terms, if converted into decimal fractions and actually added, will produce the value $a = 2.71828182845904523536028$, for which the last digit is correct.[8] Logarithms are already constructed for this base, [and] are usually called natural logarithms or hyperbolic logarithms, since the quadrature of the hyperbola[9] can be expressed by such a logarithm. But for the sake of brevity we will consistently put for this number $2.718281828459\ldots$ the letter e, which will thus denote the base of the natural or hyperbolic logarithms, ...

Task 5 Why do you think Euler chose a "so that $k = 1$" in his series (3)?

We can obtain an expression for Euler's special a value as the limit of a *sequence*, and then use modern methods with Euler's ideas to prove this sequence converges. To justify Euler's work from a modern point of view, let's look at the key equation (2)

$$(1 + kz/j)^j = 1 + \frac{1}{1}kz + \frac{1\,(j-1)}{1 \cdot 2 \cdot j}k^2z^2 + \frac{1\,(j-1)\,(j-2)}{1 \cdot 2j \cdot 3j}k^3z^3 + \cdots$$

and set $k = 1, z = 1$ as Euler did, but suppose j is a *natural number.*

Task 6 Apply the finite Binomial Theorem for natural number j to expand $(1 + 1/j)^j$ as a finite series and show that

$$(1 + 1/j)^j = 1 + \frac{1}{1} + \frac{1\,(j-1)}{1 \cdot 2 \cdot j} + \frac{1\,(j-1)\,(j-2)}{1 \cdot 2j \cdot 3j} + \cdots + \frac{1\,(j-1)\,(j-2)\cdots(j-(j-1))}{1 \cdot 2j \cdot 3j \cdots (jj)}.$$

[8] *Translator's Note*: This was Euler's way of saying that he has computed an approximation of e that is correct to 23 decimal places — all without a calculator!

[9] The phrase "quadrature of the hyperbola" refers to the area of a region between the x-axis and hyperbola $y = 1/x$.

Now we form a sequence $(c_j)_{j=1}^{\infty}$ with $c_j = (1 + 1/j)^j$. If we can justify taking the limit of this sequence, we should obtain Euler's number e.

Task 7 Show that

$$c_j = 1+\frac{1}{1}+\frac{(1-1/j)}{1\cdot 2}+\frac{(1-1/j)\,(1-2/j)}{1\cdot 2\cdot 3}+\cdots+\frac{(1-1/j)\,(1-2/j)\cdots(1-(j-1)\,/j)}{1\cdot 2\cdot 3\cdot\cdots\cdot j}.$$

Task 8 Write out c_{j+1} and compare it term by term with c_j. What can you conclude about the sequence $(c_j)_{j=1}^{\infty}$?

Task 9 Compare c_j to a geometric series to show the sequence (c_j) is bounded.

Hints: Use the series form for c_j you found in Task 7, and compare it term by term with a geometric series.

Task 10 Apply the Monotone Convergence Theorem to give a modern proof that $\lim_{j\to\infty} (1 + 1/j)^j$ exists.

We now have a modern justification that the constant e can be defined as $e = \lim_{j\to\infty} (1 + 1/j)^j$. You may recall from introductory calculus courses that Euler was correct with the series expansion (4) for e. A modern justification of the series expansion for e is beyond the scope of this project.[10]

Task 11 This task should give you some appreciation for Euler's series (4) when he wanted to find a good decimal approximation for e. Remember that he had no computers at his disposal!

(a) Use technology to find both $c_4 = (1 + 1/4)^4$ and the partial sum $P_4 = \sum_{n=0}^{4} \frac{1}{n!}$.

(b) How close are c_4 and P_4 to Euler's Section 122 decimal approximation to e, and which is more accurate?

(c) Use technology to find a value of j for which c_j is closer to e than P_4.

Task 12 Euler wanted to use his work to express the function e^z as an infinite series. Use Euler's (2) and his Section 116 infinitesimal methods to find an infinite series expression for e^z. How does this series compare with the Taylor series for e^z you learned about in introductory calculus?

[10] Augustin-Louis Cauchy (1789–1857) proved the convergence of this series expansion for e in his 1821 textbook *Cours d'Analyse* (*Course on Analysis*). The details of Cauchy's proof, along with a more thorough exploration of today's methods for studying infinite series, can be found in the project "Abel and Cauchy on a Rigorous Approach to Infinite Series," [Ruch, 2017].

Recall (from Euler's Section 114) that Euler was investigating how the number k depends on the logarithm base a, where $\omega = \log_a(1 + k\omega)$ with "infinitely small" ω. We can re-interpret this equation without reference to infinitesimals in terms of the following limit:

$$\lim_{\omega \to 0} \frac{\omega}{\log_a(1 + k\omega)}. \tag{5}$$

The final task below uses this limit to reinforce the connection between the values of a and k.

Task 13 As you may recall, the derivative of $\log_a x$ is $\dfrac{1}{x \ln a}$. You will need this formula for part (a).

(a) Use introductory calculus techniques to find the limit (5) in terms of a and k.

(b) Since $\omega = \log_a(1 + k\omega)$ with "infinitely small" ω, explain why this limit (5) should be 1 for any pair a, k where k is chosen to depend properly on a.

(c) In particular, when $a = e$, what value of k is required for limit (5) to be 1?

(d) Use limit (5) to find the **exact** values of k when $a = 10$ and $a = 2$. Use your answers to reflect on the decimal values for k and the graphs of $y = a^\omega$, $y = 1+k\omega$ that you found in Task 1 and Task 2.

References

J. Bernoulli. Principia calculi exponentialium seu percurrentium (Principles of a calculus of exponential and percurrent quantities). *Acta Eruditorium*, March: 125–133, 1697.

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L. Euler. De la controverse entre Mrs. Leibniz et Bernoulli sur les logarithmes des nombres negatifs et imaginaires (On the controversy between Mister Leibniz and Mister Bernoulli about the logarithms of negative and imaginary numbers). *Mémoires de l'académie des sciences de Berlin*, 5:139–179, 1749.

D. Ruch. Abel and Cauchy on a Rigorous Approach to Infinite Series. 2017. Primary Source Project available at `https://digitalcommons.ursinus.edu/triumphs_analysis/4/`.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to be used in a course on real analysis or foundations of the real numbers. Specifically, its content goals are to:

1. Rigorously prove that the sequence $\{(1+1/j)^j\}$ converges to e by modernizing Euler's proof.

2. Develop Euler's idea of e as an optimal logarithm base.

3. Understand the relationship between the series and sequence expressions for e, as developed by Euler.

Student Prerequisites

The project is written for a course in Real Analysis with the assumption that students have studied sequences and are familiar with the Monotone Convergence Theorem. If students are rusty with the Binomial Theorem or the derivative of $\log_a(x)$, some quick "Just in Time" review will be needed.

PSP Design and Task Commentary

The heart of this project for an introductory real analysis course is giving a modern justification of $e = \lim\limits_{j\to\infty}(1+1/j)^j$ using Euler's ideas along with some modern theory. The approach using the Monotone Convergence Theorem, as outlined in Tasks 6–10, is a common approach in current analysis textbooks. Reading about it in Euler's own words gives context to the exercises and some appreciation of his dexterity with infinitesimals and series, as well as the close connection with e as a logarithm base to motivate the definition. This series development of e^z is an interesting alternative to the Taylor series approach students have seen in introductory calculus courses.

One question for instructors and students alike is how formally and thoroughly to treat Euler's manipulations of infinitely large and small numbers. The project author is of the opinion that students already have a personal sense of what these objects are and how they should work, having been through introductory calculus courses. Euler made a good case for his development in the passages quoted in the project so students can follow his reasoning. Since this project is designed for an introductory real analysis course, a lengthy discussion of infinitesimal calculus is not appropriate. However, instructors for other courses may want to spend more time on these issues.

In Tasks 1 and 2, it is interesting to note that if students try to approximate k better by using smaller values of $k\omega$, they may run into technology problems. For example, a TI-84 calculator evaluates $\dfrac{10^{-10}}{\log_{10}(1+10^{-10})}$ to be 2.302585093, but the Mathematica 10 computer algebra system does not fare so well, producing 2.30258490259. This is likely the case because the TI calculator uses base 10 floating point arithmetic, while Mathematica uses base 2. Using $k\omega = 10^{-6}$ accomplishes the main goal while avoiding technology problems. Students revisit these k values in the last exercise of the project.

Suggestions for Classroom Implementation

This project takes around two 75-minute class sessions plus homework using the following methodology, very similar to David Pengelley's "A, B, C" method described on his website.[11]

1. Students do some advance reading and light preparatory exercises before each class. This should be counted as part of the project grade to ensure students take it seriously. Be careful not to get carried away with the exercises or your grading load will get out of hand! Some instructors have students write questions or summaries based on the reading.

2. Class time is largely dedicated to students working in groups on the project — reading the material and working exercises. As they work through the project, the instructor circulates through the groups asking questions and giving hints or explanations as needed. Occasional student presentations may be appropriate. Occasional full class guided discussions may be appropriate, particularly for the beginning and end of class, and for difficult sections of the project. I have found that a "participation" grade suffices for this component of the student work. Some instructors collect the work. If a student misses class, I have them write up solutions to the exercises they missed. This is usually a good incentive not to miss class!

3. Some exercises are assigned for students to do and write up outside of class. Careful grading of these exercises is very useful, both to students and faculty. The time spent grading can replace time an instructor might otherwise spend preparing for a lecture.

Sample Implementation Schedule (based on a 75-minute class period)

Full implementation of the project can be accomplished in two 75-minute class sessions, as outlined below.

- Day 1. Assign through Task 1 as advance preparation work; complete Tasks 2–6 in-class.
- Day 2. Assign Task 7 as advance preparation work; complete Tasks 8, 10, 13 in-class.
- Homework. Tasks 9, 11, 12.

Connections to other Primary Source Projects

The following additional projects based on primary sources are also freely available for use in an introductory real analysis course; the PSP author name for each is listed parenthetically, along with the project topic if this is not evident from the PSP title. Shorter PSPs that can be completed in at most 2 class periods are designated with an asterisk (*). Classroom-ready versions of the last two projects listed can be downloaded from `https://digitalcommons.ursinus.edu/triumphs\_topology`; all other listed projects are available at `https://digitalcommons.ursinus.edu/triumphs\_analysis`.

- *Why be so Critical? 19th Century Mathematics and the Origins of Analysis** (Janet Heine Barnett)

[11] `https://web.nmsu.edu/~davidp/`

- *Investigations into Bolzano's Bounded Set Theorem* (David Ruch)
- *Stitching Dedekind Cuts to Construct the Real Numbers* (Michael Saclolo)
  Also suitable for use in an Introduction to Proofs course.
- *Investigations Into d'Alembert's Definition of Limit** (David Ruch)
  A second version of this project suitable for use in a Calculus 2 course is also available.
- *Bolzano on Continuity and the Intermediate Value Theorem* (David Ruch)
- *Understanding Compactness: Early Work, Uniform Continuity to the Heine-Borel Theorem* (Naveen Somasunderam)
- *An Introduction to a Rigorous Definition of Derivative* (David Ruch)
- *Rigorous Debates over Debatable Rigor: Monster Functions in Real* (Janet Heine Barnett; properties of derivatives, Intermediate Value Property)
- *The Mean Value Theorem*(David Ruch)
- *The Definite Integrals of Cauchy and Riemann* (David Ruch)
- *Henri Lebesgue and the Development of the Integral Concept** (Janet Heine Barnett)
- *Abel and Cauchy on a Rigorous Approach to Infinite Series* (David Ruch)
- *The Cantor set before Cantor** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.
- *Topology from Analysis** (Nicholas A. Scoville)
  Also suitable for use in a course on topology.

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