CLARK UNIVERSITY 1887

Jacobians

Math 131 Multivariate Calculus

D Joyce, Spring 2014

Jacobians for change of variables. Since double integrals are iterated integrals, we can use the usual substitution method when we're only working with one variable at a time. But there's also a way to substitute pairs of variables at the same time, called a change of variables. Some integrals can be evaluated most easily by change of variables. In particular, changing to polar coordinates is often helpful.

If the original variables are (x, y), and the new variables are (u, v), then there's a function $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2$ that gives u and v in terms of x and y, that is, $\mathbf{T}(u, v) = (x(u, v), y(u, v)$. We'll see that we need something called the Jacobian, denoted $\dfrac{\partial(x, y)}{\partial(u, v)}$, to effect a change of variables in double integrals.

First, we'll review ordinary substitution for single variables to see what we're generalizing. Second, we'll look at a change of variables in the special case where that change is effected by a linear transformation $\mathbf{T} : \mathbf{R}^2 \to \mathbf{R}^2$. Finally, we'll look at the general case where T doesn't have to be linear.

Recall the change of variables for single integrals. Let's start with an indefinite integral

$$\int f(x)\,dx$$

and apply a substitution $x = x(u)$. Note that we're using x both as a variable and a function, but if you prefer, you can use a different symbol for the function. After substitution, we get the integral

$$\int f(x)\,dx = \int f(x(u))\,\frac{dx}{du}\,du.$$

Now let's add limits of integration. If the limits of integration for u are a and b, then the limits of integration for x will be $x(a)$ and $x(b)$.

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\,\frac{dx}{du}\,du.$$

We want to generalize this to multiple integrals.

Before we do, however, let's change the interval of integration into a domain of integration since when we generalize to two variables, we'll be talking about domains. Let D^* denote the interval $[a, b]$, which is a subset of $\mathbf{R}$, and let $D = [x(a), x(b)]$ be the image of that interval. Then the rule for substitution becomes

$$\int_D f(x)\,dx = \int_{D^*} f(x(u))\,\frac{dx}{du}\,du.$$

Actually, this isn't always valid, since when we change from intervals to domains, we lose the orientation of the interval. Take, for instance, the integral $\displaystyle\int_0^1 -x\,dx$, and apply the substitution $x = -u$, $dx = -du$. With limits on our intervals, we get

$$\int_0^1 -x\,dx = \int_0^{-1} -u\,(-du),$$

which is correct, since both integrals equal $-\frac{1}{2}$. But, in terms of domains of integration, we have

$$\int_{[0,1]} -x\,dx = \int_{[-1,0]} -u\,(-du),$$

which is wrong, because the right integral means $\displaystyle\int_{u=-1}^{0} -u\,(-du)$. The problem is that domains are subsets without orientation. Thus, the correct rule for substitution when using domains has absolute values of the derivative:

$$\int_D f(x)\,dx = \int_{D^*} f(x(u))\,\left|\frac{dx}{du}\right|\,du,$$

and that's the form we'll be generalizing.

Linear transformations. Consider a linear transformation $\mathbf{T}: \mathbf{R}^2 \to \mathbf{R}^2$. Such a linear transformation can be described by a 2×2 matrix A. We identify ordered pairs as column vectors, so $(x, y) \in \mathbf{R}^2$ is identified with $\begin{bmatrix} x \\ y \end{bmatrix}$, and $(u, v) \in \mathbf{R}^2$ is identified with $\begin{bmatrix} u \\ v \end{bmatrix}$. Then the equation

$$(x, y) = \mathbf{T}(u, v) = (au + bv, cu + dv),$$

becomes the matrix equation

$$\mathbf{T}(u, v) = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{T}(u, v) = A \begin{bmatrix} u \\ v \end{bmatrix}$$

where A is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Note that the entries of the matrix A which describes the linear transformation $\mathbf{t}$ are actually partial derivatives of $\mathbf{T}$.

$$A = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

A matrix A sends the unit square (the square with two sides being the standard unit vectors $\mathbf{i}$ and $\mathbf{j}$) to a parallelogram with two sides being the columns of A, namely, $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$. The area of this parallelogram is $|\det(A)|$, the absolute value of the determinant of A. More generally, if D^* is any region in $\mathbf{R}^2$, and $D = \mathbf{T}(D^*)$ is its image under this linear transformation, then the area of D is $|\det(A)|$ times the area of D^*.

Now, let's look at double integrals. First consider the case when the integrand is the constant 1. Then $\iint_D 1\,dx\,dy$ equals the area of D. We can rewrite the final statement in the last paragraph

$$\text{Area}(D) = |\det(A)| \text{ Area}(D^*),$$

in terms of integrals as

$$\iint_D 1\,dx\,dy = \iint_{D^*} 1\,|\det(A)|\,du\,dv.$$

Change of variables for double integrals. We have to make two generalizations to make that last equation into a rule for change of variables in double integrals. First, the integrand has to be changed from the constant 1 to a general scalar-valued function f. Second, the transformation $\mathbf{T}$ has be generalized from a linear transformation to a nonlinear transformation $\mathbf{T}$.

Let's replace the constant integrand 1 by a function $f: \mathbf{R}^2 \to \mathbf{R}$. We'll simply get

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(\mathbf{T}(u, v))|\det(A)|\,du\,dv.$$

Here, $\mathbf{T}(u, v) = (x(u, v), y(u, v))$. Again, we're treating x and y as both variables and functions.

The justification for this generalization is that the solids whose volumes the double integrals describe are being stretched/squeezed from the (u, v)-plane to the (x, y)-plane, but their heights, which are given by f, aren't being changed at all.

Next, let $\mathbf{T}$ be any transformation $\mathbf{R}^2 \to \mathbf{R}^2$, not necessarily a linear transformation. (Although $\mathbf{T}$ is a vector-valued function, and, in fact, it's a vector field, we'll call it a transformation because we're treating it in a different way.)

The matrix A of partial derivatives (which is a constant matrix when $\mathbf{T}$ is a linear transformation) has a determinant which is called the *Jacobian* of $\mathbf{T}$ and denoted

$$D\mathbf{T}(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

Although $\mathbf{T}$ is not a linear transformation, this Jacobian describes the stretching/squeezing at particular points, and so the general change of variables has the same equation.

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(\mathbf{T}(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv,$$

where $\left| \dfrac{\partial(x, y)}{\partial(u, v)} \right|$ denotes the absolute value of the Jacobian.

Examples of change of variables in double integrals. Determine the value of

$$\iint_D \sqrt{\frac{x+y}{x-2y}}\, dA$$

where D is the region in $\mathbf{R}^2$ enclosed by the lines $y = x/2$, $y = 0$, and $x + y = 1$.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`