Change of Variables and the Jacobian

Prerequisite: Section 3.1, Introduction to Determinants

In this section, we show how the determinant of a matrix is used to perform a change of variables in a double or triple integral. This technique generalizes to a change of variables in higher dimensions as well. Although the prerequisite for this section is listed as Section 3.1, we will also need the fact that $|\mathbf{A}| = |\mathbf{A}^T|$ from Section 3.3.

▶ Substitution in One Variable

The following example serves to recall the method of integration by substitution from calculus:

EXAMPLE 1 To compute $\int_1^5 \sqrt{3x+1}\,dx$, we first make the substitution $u = 3x+1$. Then $du = 3\,dx$, and so

$$\begin{aligned}\int_1^5 \sqrt{3x+1}\,dx &= \frac{1}{3}\int_1^5 \sqrt{3x+1}\,(3\,dx) = \frac{1}{3}\int_4^{16} \sqrt{u}\;du \\ &= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}\bigg|_4^{16} = \frac{2}{9}(16^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{2}{9}(64-8) = \frac{112}{9}.\end{aligned}$$

Note the factor of 3 in $du = 3\,dx$. This indicates that the variable u covers 3 units of distance for each single unit of x. (It is as if u is measured in feet, while x is measured in yards.) Note that the length of the x-interval is only 4 units (from 1 to 5), while the length of the u-interval is 12 units (from 4 to 16). The factor of 3 in the du term compensates for this change. ■

In Example 1, the substitution variable u is a linear function of x, and so the change in units is constant throughout the given interval. In the next example, however, the substitution is non-linear.

EXAMPLE 2 Consider $\int_1^2 \frac{2x}{(x^2+1)^2}\,dx$. Let $u = x^2+1$. Then $du = 2x\,dx$. The integral is then calculated as

$$\int_1^2 \frac{2x}{(x^2+1)^2}\,dx = \int_2^5 \frac{du}{u^2} = -\frac{1}{u}\bigg|_2^5 = -\frac{1}{5} - \left(-\frac{1}{2}\right) = \frac{3}{10}.$$

The factor $2x$ in $du = 2x\,dx$ indicates that the unit conversion from x to u is not constant. As the x-interval $[1,2]$ is stretched into the u-interval $[2,5]$, the stretching is done unevenly. For example, at $x=1$, the scaling factor $2x = 2(1) = 2$, and so at this point, the length of a u unit is 2 times smaller than the length of an x unit. However, at $x = 1.5$, the scaling factor $2x = 2(1.5) = 3$, and so at this point, a u unit is 3 times smaller than an x unit.

In particular, the x-interval $[1.5, 1.51]$ (of length 0.01) is mapped to the u-interval $[3.25, 3.2801]$ (having length 0.0301). That is, the u-interval is approximately 3 times as long, because the scaling factor is 3 at $x = 1.5$. The error in using 3 as the scaling factor in this case is 0.0001, or $0.3\overline{3}\%$. As the length of the x-interval approaches 0, as it would in computing Riemann sums for integrals, the percent error in the scaling factor also approaches 0. ■

In general, since $\frac{du}{dx}$ is the rate of change of u with respect to x, its presence in the formula $du = \frac{du}{dx}\,dx$ keeps track of the amount of stretching involved in converting from x-coordinates to u-coordinates. Thus, $\frac{du}{dx}$ is the desired scaling factor for a change of variable in single-variable integration.

► Double Integrals

We now consider the analogous situation using two variables.

Example 3 The area of the parallelogram P indicated in Figure 1 is given by the following double integral:

$$\text{Area } = \iint_P 1\,dx\,dy.$$

Converting this double integral into an iterated integral would be tedious. However, we can compute the area of P using Theorem 3.1. The vectors $\mathbf{w}_1 = [2,1]$ and $\mathbf{w}_2 = [-1,1]$ correspond to the sides of P, and so

$$\text{area of } P = \text{absolute value of } \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = |2-(-1)| = 3.$$

Let us now examine the effect of a change of variables on the area. Since the sides of P are the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, we first create new variables u and v to satisfy the equation

$$[x,y] = u\mathbf{w}_1 + v\mathbf{w}_2 + [1,1] = u[2,1] + v[-1,1] + [1,1];$$

that is, $x = 2u - v + 1$, $y = u + v + 1$. Then, (x,y) vertices correspond to (u,v) vertices as follows:

(x,y)	(u,v)
$(1,1)$	$(0,0)$
$(0,2)$	$(0,1)$
$(3,2)$	$(1,0)$
$(2,3)$	$(1,1)$

Thus, in converting to the (u,v) coordinate system, the parallelogram P is mapped to the unit square S shown in Figure 2. Therefore, it follows that

$$\iint_S 1\,du\,dv = \text{area of } S = 1.$$

Since the parallelogram P does not have area 1, we must be missing a scaling factor of the type seen in the single variable case. Note that the scaling factor must be constant in this case, as in Example 1, because the change of coordinates involves only linear functions. Since the area of $P = 3(\text{area of } S)$, the scaling factor must be precisely 3. ■

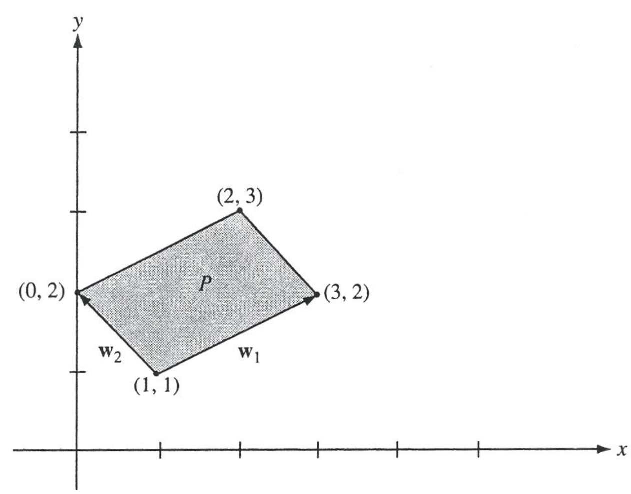

Figure 1: The parallelogram in the (x,y) system with vertices $(1,1),(0,2),(3,2),(2,3)$

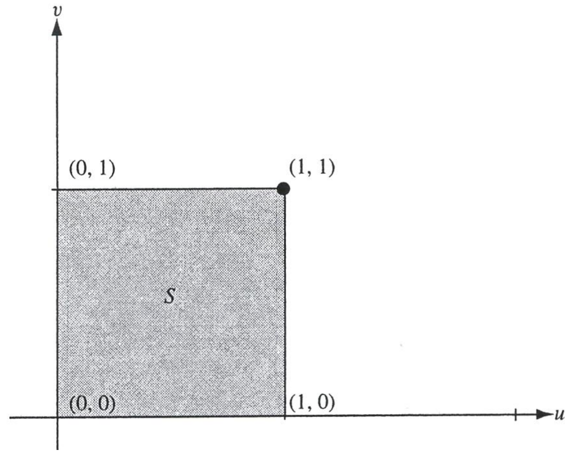

Figure 2: The square in the (u, v) system with vertices $(0, 0), (0, 1), (1, 0), (1, 1)$

Note in Example 3 that we can work backwards to compute the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ from the formulas for x and y as $\mathbf{w}_1 = \left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right]$, and $\mathbf{w}_2 = \left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right]$. This will work in general for all change of variable transformations. The idea behind this is that a unit rectangle in (u, v) coordinates is mapped to a region in (x, y) coordinates that is approximated by a parallelogram whose sides are $\mathbf{w}_1$ and $\mathbf{w}_2$, as in Figure 3. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are tangent to the curved boundary of the actual image of the rectangle under the transformation. But differentiation, along with finding the tangent direction, also measures the rate of change, and so the lengths of $\mathbf{w}_1$ and $\mathbf{w}_2$ also represent the amount of stretching taking place in each of these directions. Hence, the scaling factor needed for the change of variable is the area of this approximating parallelogram, which, by Theorem 3.1, is the absolute value of $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$.

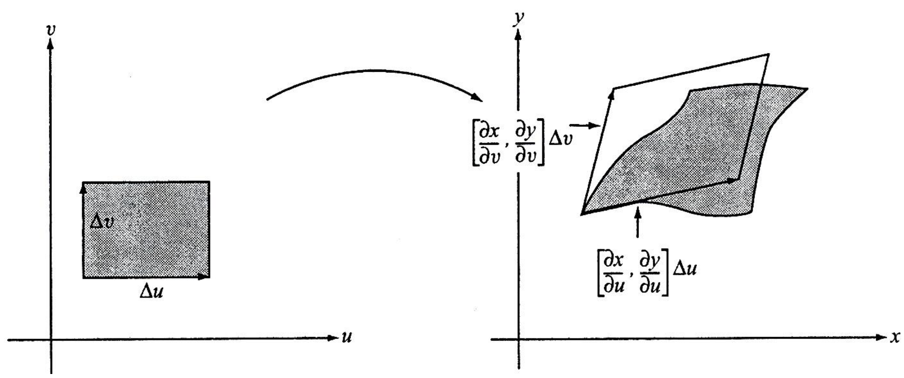

Figure 3: Converting a rectangle in (u, v) coordinates to an approximate parallelogram in (x, y) coordinates

In Section 3.3, it is proved that for any square matrix $\mathbf{A}$, $|\mathbf{A}| = |\mathbf{A}^T|$. Hence we could have also found the scaling factor as the absolute value of $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ instead. The matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

is called the **Jacobian matrix** of the change of coordinates function $\begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$. We will refer to $|\mathbf{J}|$ as the **Jacobian determinant**. In general, the correct scaling factor to change an integral $\iint\limits_R f(x,y)\,dx\,dy$ over a region R into (u,v) coordinates is the absolute value of the Jacobian determinant, that is, $\left||\mathbf{J}|\right|$. Therefore, if S is the region in (u,v) coordinates that corresponds to R, then

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_S f(x(u,v),y(u,v))\,\left||\mathbf{J}|\right|\;du\,dv.$$

Just as in the one-variable case, the scaling factor can vary if the change of coordinates is nonlinear, as we will see shortly.

Polar Coordinates

The polar coordinate system is frequently used to represent points in a two-dimensional space. In polar coordinates, each point $P = (x,y)$ in the plane is assigned a pair[1] of coordinates (r,θ), where r is the distance from the origin to P, and θ is the angle between the positive x-axis and the vector having initial point at the origin and terminal point P (see Figure 4). In all quadrants, the transformation from polar coordinates to standard (rectangular) coordinates is given by $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. We can also convert from rectangular coordinates to polar coordinates using $\begin{cases} r^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x} \quad (\text{when } x \neq 0) \end{cases}$.

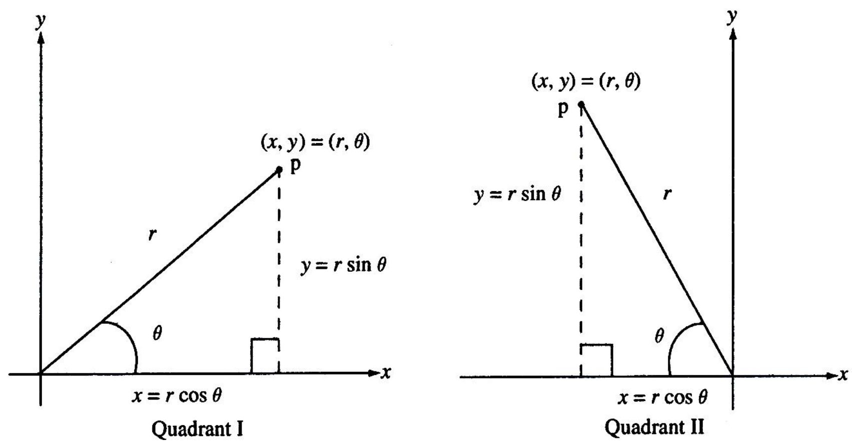

Figure 4 Relationship between standard coordinates and polar coordinates in Quadrants I and II

It is useful to express certain double integrals in polar coordinates if the region of integration (and/or the function involved) has radial or angular symmetry. In these instances, we need to compute the determinant of the Jacobian matrix in order to include the proper scaling factor when we change coordinates.

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

[1] The assignment of polar coordinates to a given point (x,y) is not unique. For example, $(x,y) = \left(\sqrt{3},1\right)$ in rectangular coordinates can be represented as (r,θ) in polar coordinates as $(2,\frac{\pi}{6})$, $(2,\frac{13\pi}{6})$, or $(-2,\frac{7\pi}{6})$. In general, $\left(\sqrt{3},1\right)$ can be expressed in polar coordinates as (r,θ), where $r = \pm\sqrt{(\sqrt{3})^2+1^2} = \pm 2$, and $\theta = \frac{\pi}{6} + k\pi$, where k is an even integer when r is positive, and k is an odd integer when r is negative.

If we are careful to ensure that $r \geq 0$, the absolute value of $|\mathbf{J}|$ is also r, and so this is our scaling factor. Hence,

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(r,\theta), y(r,\theta))\,r\,dr\,d\theta,$$

where R^* is the region in the polar coordinate system corresponding to R. The next example illustrates this geometrically.

Example 4 Consider the square S in the (r,θ) (polar) coordinate system with left bottom corner at $(2, \frac{\pi}{6})$, width $\Delta r = 0.1$, and height $\Delta\theta = 0.1$. The image R of this square in the (x,y) system under the polar coordinate map $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ is shown in Figure 5.

Now, the square S has area $\Delta r\Delta\theta = (0.1)(0.1) = 0.01$, and thus the area of R is approximately equal to the product of the Jacobian determinant, $r = 2$, with the area of S. Hence, the area of $R \approx 2(0.01) = 0.02$.

To understand this approximation, recall that the columns of the Jacobian matrix represent vectors tangent at the corner point to the curved edges of R. When these vectors are scaled properly by multiplying by Δr and $\Delta\theta$, respectively, they represent the sides of a parallelogram (shown in Figure 6) whose area approximates the area of R. (In this particular case, the dot product of the columns is zero, and so the parallelogram is a rectangle.)

Finally, we compute the actual area of R for comparison purposes. The actual area of R is $\frac{\Delta\theta}{2\pi}$ (the portion of the circle involved) times the differences of the areas of the circles of radii 2.1 and 2.0. Hence,

$$\text{area of } R = \frac{\Delta\theta}{2\pi}(\pi(2.1^2) - \pi(2^2)) = \frac{0.1}{2\pi}(\pi(0.41)) = \frac{0.041}{2} = 0.0205.$$

Thus, in this case, the scale factor obtained from the Jacobian induces an error of only 0.0005, or, 2.5%. Of course, in the actual integration, both $\Delta r \to 0$ and $\Delta\theta \to 0$, which makes the percent error approach 0 as well (although we do not prove this here). ■

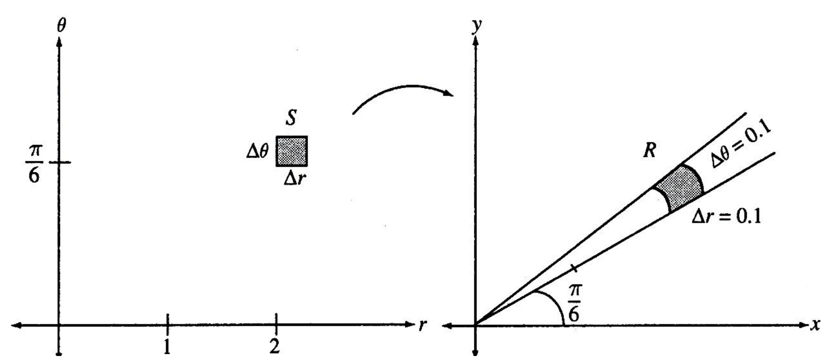

Figure 5: Image R of polar coordinate system square S in rectangular coordinates

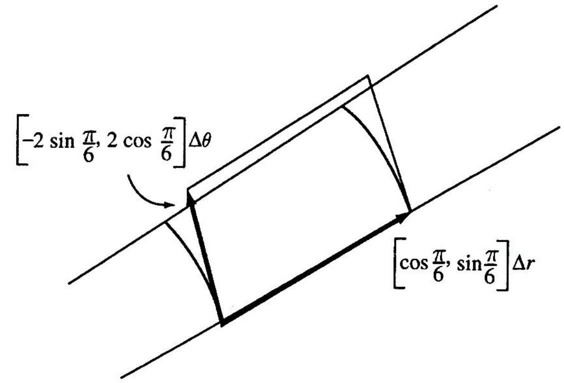

Figure 6 The parallelogram formed by the columns of the Jacobian at the point $(2, \frac{\pi}{6})$

EXAMPLE 5 Consider $\iint\limits_R \sqrt{x^2+y^2}\,dx\,dy$ over the region R given by $0 \le r \le 1+\cos\theta$ in polar coordinates (see Figure 7). Now, $\sqrt{x^2+y^2} = r$, and so

$$\begin{aligned}
\iint\limits_R \sqrt{x^2+y^2}\,dx\,dy &= \iint\limits_R r\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\right)\Bigg|_0^{1+\cos\theta} d\theta = \frac{1}{3}\int_0^{2\pi}(1+\cos\theta)^3\,d\theta \\
&= \frac{1}{3}\int_0^{2\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\,d\theta \\
&= \frac{1}{3}\int_0^{2\pi}(3\cos\theta+\cos^3\theta)\,d\theta + \frac{1}{3}\int_0^{2\pi}(1+3\cos^2\theta)\,d\theta.
\end{aligned}$$

An appeal to symmetry considerations (or a tedious computation) shows the first integral equals 0. Using a double-angle formula on the second integral, we obtain

$$\frac{1}{3}\int_0^{2\pi}\left(1+3\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)\right)d\theta = \left(\frac{5}{6}\theta+\frac{1}{4}\sin 2\theta\right)\Bigg|_0^{2\pi} = \frac{5\pi}{3}.$$

■

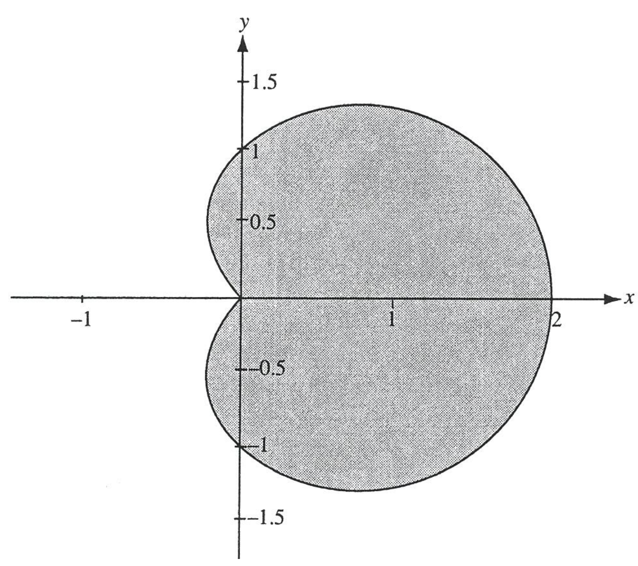

Figure 7: The region R in polar coordinates given by $0 \le r \le 1+\cos\theta$

▶ Triple Integrals

The situation for change of variables in three dimensions is similar. When converting an integral in (x, y, z) coordinates to an integral in (u, v, w) coordinates, any rectangular solid based at the point (x, y, z) and having sides Δx, Δy, and Δz is mapped to a region approximated by a parallelepiped. The sides of this parallelepiped are the columns of the Jacobian matrix evaluated at (x, y, z) multiplied by Δx, Δy, and Δz, respectively. Thus, by Theorem 3.1, the absolute value of the Jacobian determinant

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

provides the correct scaling factor for converting from xyz-space to uvw-space. That is, $dx\,dy\,dz = \Big||\mathbf{J}|\Big|\,du\,dv\,dw$.

▶ Spherical Coordinates

One coordinate system frequently used in three dimensions is spherical coordinates. If $P = (x, y, z)$ is a point in the rectangular coordinate system and $\mathbf{v}$ is a vector from the origin to P, then P is assigned coordinates (ρ, ϕ, θ) in spherical coordinates, where $\rho = ||\mathbf{v}||$, ϕ is the angle between the vector $[0, 0, 1]$ and $\mathbf{v}$, and θ is the angle between the vector $[1, 0, 0]$ and the projection of $\mathbf{v}$ onto the xy-plane (see Figure 8). From elementary trigonometry, we find that

$$\begin{array}{ll} x = \rho \sin\phi \cos\theta & \rho^2 = x^2 + y^2 + z^2 \\ y = \rho \sin\phi \sin\theta & \tan\theta = \frac{y}{x}, \text{ when } x \neq 0 \\ z = \rho \cos\phi & \cos\phi = \frac{z}{\sqrt{x^2+y^2+z^2}}. \end{array}$$

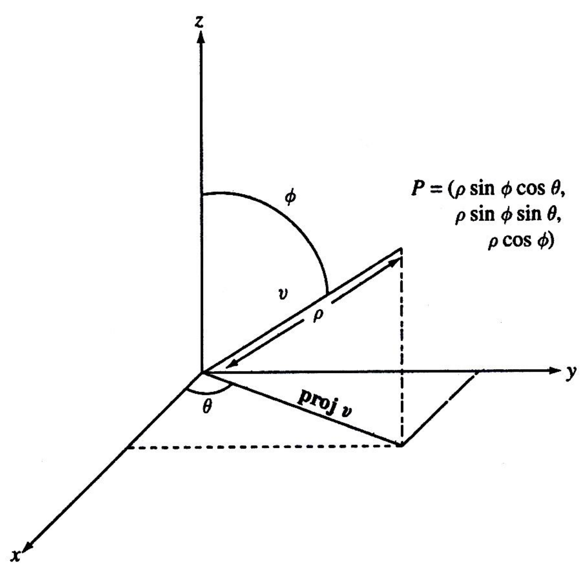

Figure 8: Spherical coordinates for $P = (x, y, z)$

Hence,

$$
\begin{aligned}
|\mathbf{J}| &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} - (-\rho\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \cos\phi(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta) \\
&\qquad + \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta) \\
&= \rho^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + \rho^2\sin^3\phi(\cos^2\theta + \sin^2\theta) \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
&= \rho^2\sin\phi.
\end{aligned}
$$

Since $0 \le \phi \le \pi$ in spherical coordinates, the quantity $\rho^2\sin\phi$ is always nonnegative. Hence, when converting an integral from xyz-coordinates to $\rho\phi\theta$-coordinates, we have

$$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example 6 We find the volume of the region R bounded below by the upper half of the cone $z^2 = x^2 + y^2$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$ (see Figure 9). Now,

$$\text{volume of } R = \iiint_R 1\,dx\,dy\,dz.$$

Converting to spherical coordinates, we have

$$\text{volume of } R = \iiint_R \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Since the radius of the sphere is $\sqrt{8}$, ρ ranges from 0 to $\sqrt{8}$. The sides of the cone are at a 45° angle from the z-axis, and so ϕ ranges from 0 to $\frac{\pi}{4}$. Hence, changing to an iterated integral, we obtain

$$
\begin{aligned}
\text{volume of } R &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{8}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left(\frac{\rho^3}{3}\sin\phi\right)\Bigg|_0^{\sqrt{8}} d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{8\sqrt{8}}{3}\sin\phi\,d\phi\,d\theta \\
&= -\frac{8\sqrt{8}}{3}\int_0^{2\pi} (\cos\phi)\Big|_0^{\frac{\pi}{4}} d\theta \\
&= -\frac{8\sqrt{8}}{3}\int_0^{2\pi} \left(\frac{\sqrt{2}}{2} - 1\right) d\theta \\
&= -\frac{8\sqrt{8}}{3}\left(\frac{\sqrt{2}}{2} - 1\right)(2\pi) \\
&= \frac{32\pi}{3}(\sqrt{2} - 1).
\end{aligned}
$$

■

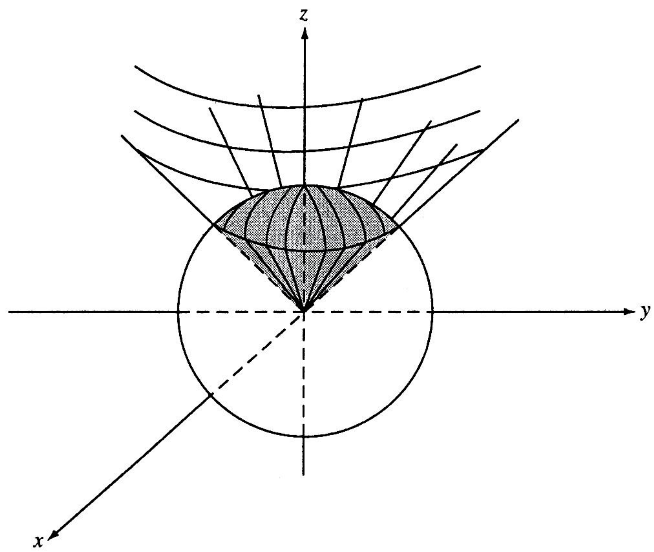

Figure 9: Region R bounded below by the upper half of the cone $z^2 = x^2 + y^2$ and bounded above by the sphere $x^2 + y^2 + z^2 = 8$

▶ Cylindrical Coordinates

Another frequently used three-dimensional coordinate system is cylindrical coordinates, (r, θ, z), in which the r and θ variables provide a polar coordinate system in the xy-plane, and z is unchanged from rectangular coordinates. Thus,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \ . \\ z &= z \end{aligned}$$

In Exercise 3, you are asked to show that the Jacobian determinant for a transformation from rectangular to cylindrical coordinates is r, and hence

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

▶ Higher Dimensions

The method we have shown for changing variables in double and triple integrals also works in general for multiple integrals in $\mathbb{R}^n$. In particular, to change from $x_1x_2\ldots x_n$ coordinates to $u_1u_2\ldots u_n$ coordinates, we must calculate the absolute value of the determinant of the Jacobian matrix,

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix},$$

and then we have,

$$dx_1\,dx_2\ldots dx_n = \Big||\mathbf{J}|\Big|\,du_1\,du_2\ldots du_n.$$

▶ New Vocabulary

cylindrical coordinates
Jacobian determinant
Jacobian matrix
polar coordinates
spherical coordinates

▶ Highlights

- For the change of coordinates function $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$, the Jacobian matrix is $\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$, and its determinant, $|\mathbf{J}|$, is called the Jacobian determinant.

- The scaling factor involved when converting a double integral from one set of coordinates to another is the absolute value of the Jacobian determinant. That is, if f is a function of variables x and y, R is a region in (x, y) coordinates, and S is the corresponding region in (u, v) coordinates, then

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| |\mathbf{J}| \right| du\,dv.$$

- When converting from (x, y) coordinates to (u, v) coordinates, $dx\,dy = \left||\mathbf{J}|\right| du\,dv$. In particular, in polar coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$, we have $dx\,dy = r\,dr\,d\theta$.

- When converting an integral in (x, y, z) coordinates to an integral in (u, v, w) coordinates, the absolute value of the Jacobian determinant

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

provides the correct scaling factor for converting from xyz-space to uvw-space. That is, $dx\,dy\,dz = \left||\mathbf{J}|\right| du\,dv\,dw$.

- In spherical coordinates, where $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, we have $dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

- In cylindrical coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we have $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

- When converting from $x_1x_2\dots x_n$ coordinates to $u_1u_2\dots u_n$ coordinates, the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \cdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{bmatrix},$$

and we have $dx_1\,dx_2\dots dx_n = \left||\mathbf{J}|\right| du_1\,du_2\dots du_n$.

▶ EXERCISES

1. For each change of variable formula, compute $dx\,dy$ in terms of $du\,dv$.

★ **a)** $x = u + v$, $y = u - v$

b) $x = u^2 + v^2$, $y = u^2 - v^2$

★ **c)** $x = u^2 - v^2$, $y = 2uv$

d) $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

★ **e)** $x = \frac{2u}{(u+1)^2+v^2}$, $y = \frac{1-(u^2+v^2)}{(u+1)^2+v^2}$

2. For each change of variable formula, compute $dx\,dy\,dz$ in terms of $du\,dv\,dw$.

★ **a)** $x = u + v$, $y = v + w$, $z = w + u$

b) $x = 3u + v + w$, $y = 3v + w$, $z = w$

★ **c)** $x = \frac{u}{u^2+v^2+w^2}$, $y = \frac{v}{u^2+v^2+w^2}$, $z = \frac{w}{u^2+v^2+w^2}$

d) $x = \frac{w}{u}$, $y = u$, $z = u\cos v$ (for $u > 0$)

3. Show that $|\mathbf{J}| = r$ for the change of variables from rectangular coordinates to cylindrical coordinates.

4. Compute each of the following integrals by changing to the indicated coordinate system:

★ **a)** $\iint\limits_R (x + y)\,dx\,dy$, where R is the region in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$; polar coordinates

b) $\iint\limits_R 1\,dx\,dy$, where R is the region inside the innermost ring of the spiral $r = \theta$ in the first quadrant (see Figure 10); polar coordinates

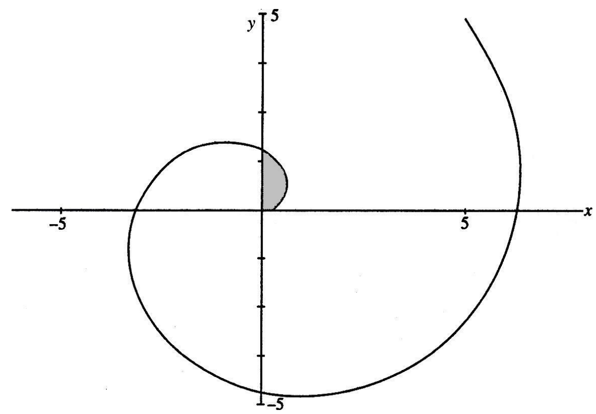

Figure 10: The spiral $r = \theta$

★ **c)** $\iiint\limits_R z\,dx\,dy\,dz$, where R is the half of the sphere of radius 1 centered at the origin which is above the xy-plane; spherical coordinates

d) $\iiint\limits_R \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$, where R is the shell between the spheres of radii 2 and 3 centered at the origin; spherical coordinates

★ **e)** $\iiint\limits_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$, where R is the region defined by $x^2 + y^2 \le 4$ and $-3 \le z \le 5$; cylindrical coordinates

★ **5.** True or False:

a) A linear change of coordinates for an integration results in a constant scaling factor with respect to the associated integrals.

b) For the change of variables $u = y$, $v = x$, we have $du\,dv = 1\,dx\,dy$.

c) A rectangle in uv-coordinates with sides Δu and Δv is mapped by a change of coordinates to a region whose area is approximated by the area of the parallelogram with sides $\left[\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right]\Delta u$ and $\left[\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right]\Delta v$.

d) The scaling factor for a change of variables in integrals is always the determinant of the Jacobian matrix.

► Answers to Selected Exercises

(1) (a) $dx\ dy = 2\ du\ dv$

(c) $dx\ dy = 4(u^2 + v^2)\ du\ dv$

(e) $dx\ dy = \left(\frac{8|v|}{((u+1)^2+v^2)^3}\right)\ du\ dv$

(2) (a) $dx\ dy\ dz = 2\ du\ dv\ dw$

(c) $dx\ dy\ dz = \left(\frac{1}{(u^2+v^2+w^2)^3}\right)\ du\ dv\ dw$

(4) (a) $\frac{52}{3}$

(c) $\frac{\pi}{4}$

(e) $\frac{800}{3}\pi$

(5) (a) T

(b) T

(c) T

(d) F